The transfer problem: A complete characterization *

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Abstract

The transfer problem is defined by the possibility for a donor country to end up better off after having given away some resources to another country. The simplest version of that problem can be formulated in a two consumer exchange economy with fixed total resources. Existence of a transfer problem at some equilibrium is known to be equivalent to instability in the case of two goods. This characterization is extended to an arbitrary number of goods by showing that a transfer problem exists at a (regular) equilibrium if and only if this equilibrium has an index value equal to -1. Samuelson's conjecture that there is no transfer problem at tatonnement stable equilibria is therefore true for any number of goods.

Keywords: *transfer problem, regular equilibrium, index value.*

JEL classification numbers: *D51, F20*

1. Introduction

Does a country's utility necessarily decrease when that country gives away some resources to another country? This problem is known in trade theory as the transfer problem and has led to a substantial literature. One aspect of the transfer problem is the characterization under simple assumptions (no trade impediments such as transportation costs and tariffs in particular) of those equilibria at which the donor country can improve its utility when giving away resources. The simplest model in which the transfer problem can be studied in the case of an arbitrary number of goods is the exchange model with two consumers and fixed total resources. See for example [13]. This question takes two very different forms depending on whether one considers the same equilibrium selection map or two different ones. The second case is equivalent to the existence of multiple equilibria. The transfer problem then has a very simple solution that is independent of the number of goods [2].

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The transfer problem for the same equilibrium selection map is much more interesting. Its study is also more difficult. A necessary condition is that the equilibrium is regular. The following results are then known: there are examples of economies that have regular equilibria with a transfer problem, i.e., such that a consumer (country) can be better off by giving away some resources [10]; there is no transfer problem at tatonnement stable equilibria ([12], footnote p.29 and [13]); tatonnement stability is not only sufficient but also necessary to prevent a transfer problem in the case of two goods [2]. Using the theory of smooth economies, I show in this paper that, at a regular equilibrium and for the equilibrium selection map defined by that equilibrium, there is a transfer problem if and only if the index value of that equilibrium is equal to -1. Utility level of the donor country then increases for any gifts that remain small enough to stay in the domain of the equilibrium selection map. This property extends to an arbitrary number of goods the characterization given in [2] for the case of two goods.

The geometric or dual formulation of the equilibrium equation in the price-income space plays a crucial role in this paper. Details can be found in [4], Chapter 5. Nevertheless, in order to make the paper as much self-contained as possible, I give a fresh and rather simple presentation of that approach, something made possible by the number of consumers that is limited to two. I also recall in this paper several concepts and definitions of the theory of smooth economies. If no prior knowledge of that theory is required, interested readers are encouraged to consult suitable references as, for example, [5, 7, 8, 11]. On the mathematical side, the part on the orientation of smooth manifolds, the orientation and intersection numbers (for two smooth submanifolds of complementary dimensions) in Chapter 3, pages 94–107, of [9] nicely complements the developments of this paper but is not strictly necessary. A broad understanding of the implicit function and inverse function theorems is all that is required on the mathematical side.

Section 2 of this paper is devoted to the main assumptions, definitions and notation. The geometric or dual formulation of the exchange model in the price-income space (limited to the case of two consumers, fixed total resources and an arbitrary number of goods) occupies Section 3. This formulation is then used in Section 4 for a complete characterization by their index value of the regular equilibria featuring a transfer problem. Concluding comments end this paper with Section 5.

2. Definitions, assumptions and notation

2.1. Goods and prices

There are $\ell \geq 2$ goods. The commodity space is $\mathbb{R}^\ell$ and $X = \mathbb{R}^\ell_{++}$ denotes the strictly positive orthant of that space. The price vector $p = (p_1, \ldots, p_\ell) \in X$ (all prices are strictly positive) is normalized by the numeraire assumption $p_\ell = 1$. Let $\bar{p} = (p_1, \ldots, p_{\ell-1}) \in \mathbb{R}^{\ell-1}_{++}$ denote the first $\ell - 1$ coordinates of the normalized price vector $p \in S$. It comes $p = (\bar{p}, 1)$. The set of numeraire normalized prices is denoted by $S = \mathbb{R}^{\ell-1}_{++} \times \{1\}$.

2.2. Consumers

There are two consumers (or countries) and a finite number ℓ of goods. Consumer i, with $1 \leq i \leq 2$, is endowed with the goods bundle $\omega_i \in \mathbb{R}^{\ell}_{++}$. The endowment vector $\omega = (\omega_1, \omega_2) \in \mathbb{R}^{\ell}_{++} \times \mathbb{R}^{\ell}_{++}$ defines an economy. Total resources, equal to the vector $r = \omega_1 + \omega_2$, are fixed. Let $\Omega = \{\omega = (\omega_1, \omega_2) \in X^2 \mid \omega_1 + \omega_2 = r\}$ denote the endowment or parameter space.

Consumer i preferences are represented by a utility function $u_i : X \to \mathbb{R}$ that satisfies the following assumptions that are standard in this kind of literature: 1) Smoothness; 2) Smooth monotonicity, i.e., $Du_i(x_i) \in X$ for $x_i \in X$ where $Du_i(x_i)$ is the gradient vector defined by the first-order derivatives of u_i; 3) Smooth strict quasi-concavity, namely, the restriction of the quadratic form defined by the Hessian matrix $D^2u_i(x_i)$ to the tangent hyperplane to the indifference surface $\{y_i \in X \mid u_i(y_i) = u_i(x_i)\}$ through x_i is negative definite; 4) The indifference surface $\{y_i \in X \mid u_i(y_i) = u_i(x_i)\}$ is closed in $\mathbb{R}^{\ell}$ for all $x_i \in X$. The utility function u_i is extended to $x_i = 0$ by setting $u_i(0) = \inf_{x_i \in X} u_i(x_i)$.

Consumer i's demand function is the map $f_i : S \times \mathbb{R}_{++} \to X$ where $f_i(p, w_i)$ is the unique solution to the problem of maximizing the utility $u_i(x_i)$ subject to the budget constraint $p \cdot x_i \leq w_i$. The demand function f_i is smooth, satisfies Walras law (namely the identity $p \cdot f_i(p, w_i) = w_i$). Its (numeraire normalized) Slutsky matrix is negative definite. (For details, see for example [5], Chapter 2.)

Consumer i's indirect utility function is defined by $\hat{u}_i(p, w_i) = u_i\big(f_i(p, w_i)\big)$.

2.3. Equilibrium

The excess demand associated with the pair $(p, \omega) \in S \times \Omega$ is the vector $z(p, \omega) \in \mathbb{R}^{\ell}$ that is equal to $z(p, \omega) = f_1(p, p \cdot \omega_1) + f_2(p, p \cdot \omega_2) - r$. The pair $(p, \omega) \in S \times \Omega$ is an equilibrium if

$$z(p, \omega) = 0, \tag{1}$$

equality known as the equilibrium equation, is satisfied. The equilibrium manifold is the subset E of $S \times \Omega$ defined by equation (1).

Let $\bar{z}(p, \omega) \in \mathbb{R}^{\ell-1}$ denote the vector defined by the first $\ell-1$ coordinates of the vector $z(p, \omega) \in \mathbb{R}^{\ell}$ for $(p, \omega) \in S \times \Omega$. It follows from the identity $p \cdot z(p, \omega) = 0$ (a consequence of Walras law satisfied by individual demand functions) that the equilibrium equation (1) is equivalent to the equation $\bar{z}(p, \omega) = 0 \in \mathbb{R}^{\ell-1}$.

2.4. Index of a regular equilibrium

The equilibrium $(p, \omega) \in E$ is regular if the $(\ell - 1) \times (\ell - 1)$ Jacobian matrix $J(p, \omega) = \dfrac{D\bar{z}}{D\bar{p}}(p, \omega)$ is invertible. By definition, the index of the regular equilibrium

$(p,\omega) \in E$ is equal to $+1$ (resp. -1) if the sign of $(-1)^{\ell-1} \det J(p,\omega)$ is > 0 (resp. < 0).

Remark 1. Roughly speaking, local stability for Walras tatonnement can be identified to the Jacobian matrix $J(p,\omega)$ having eigenvalues with strictly negative real parts. The product of these eigenvalues being equal to $\det J(p,\omega)$, a stable equilibrium always has an index value equal to $+1$. The converse is not true.

2.5. Equilibrium selection maps

Let $(p,\omega) \in E$ be a regular equilibrium. It is then possible to apply the implicit function theorem to the equation $\bar{z}\big((\bar{p},1),\omega\big) = 0$ where the unknown is the vector $\bar{p} \in \mathbb{R}^{\ell-1}_{++}$. Then, there exists a neighborhood U of ω and a neighborhood $V \subset E$ of the equilibrium $(p,\omega) \in E$ and a smooth map $s : U \to S$ such that the map $\sigma : U \to V$ defined by $\sigma(\omega') = \big(s(\omega'),\omega'\big)$ is a diffeomorphism between U and V (i.e., a smooth bijection with a smooth inverse map). For a neighborhood U of ω that is small enough, the map $\sigma : U \to V$ (resp. $s : U \to S$) depends only on the regular equilibrium $(p,\omega) \in E$. The map $\sigma : U \to V$ (resp. $s : U \to S$) is known as the local equilibrium selection map (resp. local equilibrium price selection map) associated with the regular equilibrium $(p,\omega) \in E$. For open sets U that are small enough, the maps σ and s are determined by the (regular) equilibrium $(p,\omega) \in E$. For details, see [5], Proposition 7.2.

2.6. The transfer problem

Let $\omega = (\omega_1,\omega_2)$ and $\omega' = (\omega'_1,\omega'_2)$ in Ω be two endowment vectors (or economies.) By definition, consumer 1 gives away some resources when the economy moves from ω to ω' if inequality $\omega'_1 \lneqq \omega_1$ (i.e., $\omega'_1 \leq \omega_1$ and $\omega'_1 \neq \omega_1$) is satisfied.

Definition 1. *There is a transfer problem at the regular equilibrium (p,ω) if there exists an endowment vector $\omega' = (\omega'_1,\omega'_2) \in U$ with $\omega'_1 \lneqq \omega_1$ such that*

$$u_1(f_1(s(\omega'), s(\omega')\cdot\omega'_1)) > u_1(f_1(s(\omega), s(\omega)\cdot\omega_1)), \tag{2}$$

where $s : U \to S$ is the local equilibrium price selection map associated with the regular equilibrium $(p,\omega) \in E$.

By definition, the transfer problem requires only the existence of one endowment vector ω' with $\omega'_1 \lneqq \omega_1$ such that inequality (2) is satisfied.

Remark 2. Definition 1 requires the equilibrium $(p,\omega) \in E$ to be regular. This restriction is minor since the set of regular equilibria is an open subset with full measure of the equilibrium manifold E by [3] or [5], Proposition 8.10.

3. The geometric approach to the transfer problem

3.1. The price-income space

The price-income space is the set $S \times \mathbb{R}^2_{++}$ that consists of the triplets (p, w_1, w_2) where w_1 and w_2 are the wealth of consumer 1 and 2 respectively. With total resources r fixed, the set $H(r)$ is the subset of $S \times \mathbb{R}^2_{++}$ defined by the linear equation $w_1 + w_2 = p \cdot r$. This is a hyperplane of dimension ℓ. This set is known as the ambient space. A set of coordinates for the ambient space $H(r)$ is given by the ℓ-tuple $(p_1, \ldots, p_{\ell-1}, w_1) = (\bar{p}, w_1) \in \mathbb{R}^{\ell}_{++}$. Then, w_2 is determined by the formula $w_2 = p \cdot r - w_1$.

3.2. The section manifold

Definition

By definition, the section manifold $B(r)$ is the subset of $H(r)$ consisting of the points $b = (p, w_1, w_2)$ that satisfy equation

$$f_1(p, w_1) + f_2(p, w_2) = r. \tag{3}$$

This set is a smooth submanifold of $H(r)$ of dimension $m - 1$ by Proposition 5.4.1 of [4]. Therefore, for $m = 2$, the section manifold is just a smooth curve. Here is a direct proof of that property.

The contract curve $P(r)$

In the case of two consumers, the section manifold $B(r)$ is closely related to the contract curve of the Edgeworth box, i.e., the set of Pareto optima associated with the fixed total resources r. Let $P(r)$ denote that set. A Pareto optimum then results from the maximization of the second consumer's utility $u_2(x_2)$ subject to the first consumer's utility constraint $u_1(x_1) = u_1$ with $u_1 \in [u_1(0), u_1(r)]$, the total resources being fixed and equal to $r \in \mathbb{R}^{\ell}_{++}$. Let $x(u_1) = (x_1(u_1), x_2(u_1))$ be the Pareto optimum that solves that constrained maximization problem and let $p(u_1) \in S$ denote the (numeraire normalized) price vector that supports that Pareto optimum $x(u_1)$. For $u_1(0) < u_1 < u_1(r)$, the price vector $p(u_1) \in S$ is collinear with the two gradient vectors $Du_1(x_1(u_1))$ and $Du_2(x_2(u_1))$; for u_1 equal to $u_1(0)$ (resp. $u_1(r)$), the price vector $p(u_1(0)) \in S$ is collinear with $Du_2(r)$ (resp. $p(u_1(r))$ with $Du_1(r)$). The set of Pareto optima is generated by varying consumer 1's utility level u_1 between $u_1(0)$ and $u_1(r)$. This defines a smooth curve with two end points, the allocations $(0, r)$ and $(r, 0)$.

The section manifold $B(r)$

The section manifold now comes in with the observation that the point $M(u_1) = (p(u_1), p(u_1) \cdot x_1(u_1), p(u_1) \cdot x_2(u_1))$ in the price-income space $H(r)$ belongs to

$B(r)$ and, conversely, any point of $B(r)$ is associated with a unique utility level $u_1 \in [u_1(0), u_1(r)]$ for the first consumer. The utility u_1 therefore parameterizes not only the contract curve $P(r)$ but also the section manifold $B(r)$. The section manifold $B(r)$ is therefore a smooth curve with two end points. One end point is the point $M_0 = (p(u_1(0)), 0, p(u_1(0)) \cdot r)$. The other end point is the point $M_1 = (p(u_1(r)), p(u_1(r)) \cdot r, 0)$.

By definition, the positive orientation of the curve $B(r)$ corresponds to increasing values of the parameter u_1. The curve $B(r)$ is separated by the point $M(u_1)$ into two connected pieces, the arc $\widehat{M_0\, M(u_1)}$ and the arc $\widehat{M(u_1)\, M_1}$. Equilibrium allocations belonging to the arc $\widehat{M_0\, M(u_1)}$ (resp. $\widehat{M(u_1)\, M_1}$) yield utility levels for consumer 1 lower (resp. higher) than u_1.

The derivative of the map $u_1 \to M(u_1) \in H(r)$ is denoted by $t(u_1)$. It represents a vector that is tangent to the curve $B(r)$ at the point $M(u_1)$. The direction defined by the vector $t(u_1)$ corresponds to increasing utility levels for consumer 1 along the curve $B(r)$ in a neighborhood of the point $M(u_1)$.

3.3. The budget hyperplane

The budget hyperplane $A(\omega)$ associated with the endowment vector $\omega \in \Omega$ is the subset of $H(r)$ defined by equation $w_1 = p \cdot \omega_1$ in the coordinate system $(\bar{p}, w_1)$. In what follows, only the part of the budget hyperplane $A(\omega)$ that is defined for $\bar{p} \in \mathbb{R}^{\ell-1}_{++}$ (i.e., for strictly positive prices) is considered.

3.4. Equilibrium and the intersection $B(r) \cap A(\omega)$

One sees readily that $(p, \omega) \in S \times \Omega$ is an equilibrium if and only if the point $b = (p, p \cdot \omega_1, p \cdot \omega_2) \in H(r)$ belongs to the intersection $B(r) \cap A(\omega)$.

The study of the equilibrium equation (1) is equivalent to the study of the intersection of the curve $B(r)$ with the budget hyperplane $A(\omega)$ when ω is varied in Ω. This geometric and highly visual approach has another remarkable and quite useful feature. The curve $B(r)$ captures all the non-linearities of equilibrium equation (1). In addition, the curve $B(r)$ does not depend at all on the endowment vector $\omega \in \Omega$. This feature will come handily in the study of the transfer problem.

3.5. Index of a regular equilibrium: geometric version

Let π_j be the vector in $\mathbb{R}^{\ell-1}$ with coordinates equal to 0 except for the j-th that is equal to 1. In the coordinate system $(\bar{p}, w_1)$ for $H(r)$, let $e_j(\omega) = (\pi_j, \omega_1^j)$. The (affine) hyperplane $A(\omega)$ is parallel to the vector subspace generated by the $\ell - 1$ vectors $e_1, \ldots, e_{\ell-1}$. The base $(e_1, e_2, \ldots, e_{\ell-1})$ then defines the positive orientation of $A(\omega)$.

Let $b = (p, p \cdot \omega_1, p \cdot \omega_2) \in H(r)$ be the point in the price-income space that is associated with the equilibrium $(p, \omega) \in E$. Let $u_1(p, \omega) = u_1(f_1(p, p \cdot \omega_1))$ be the

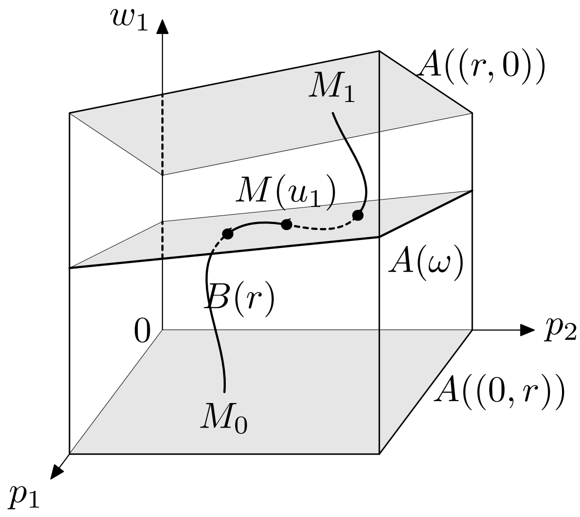

Figure 1: $A(\omega)$ and $B(r)$

utility of consumer 1 at the equilibrium allocation $x = (f_1(p, p \cdot \omega_1), f_2(p, p \cdot \omega_2))$. It comes $b = M(u_1(p, \omega))$.

Regularity of the equilibrium $(p, \omega) \in E$ is equivalent to the transversality of the smooth submanifolds $B(r)$ and $A(\omega)$ at b. The ℓ vectors $e_1(\omega)$, ..., $e_{\ell-1}(\omega)$ and $t(u_1(p, \omega))$ are then linearly independent in $H(r)$ and the determinant

$$\Delta(p, \omega) = \det(e_1(\omega), \dots, e_{\ell-1}(\omega), t(u_1(p, \omega)))$$

is $\neq 0$.

Lemma 1. *The index number of the regular equilibrium $(p, \omega) \in E$ is equal to $+1$ (resp. -1) if $\Delta(p, \omega)$ is > 0 (resp. < 0).*

Proof. It is possible to show directly, but after somewhat tedious and lengthy computations, that $\Delta(p, \omega)$ has the sign opposite to $\det \frac{D\bar{z}}{D\bar{p}}(p, \omega)$ for any regular equilibrium $(p, \omega) \in E$, which would prove the Lemma.

The following short proof avoids any computation by exploiting the connectedness of the curve $B(r)$ through its parameterization by consumer 1's utility level $u_1 \in [u_1(0), u_1(r)]$. Let $b(u_1) = (p(u_1), w_1(u_1), w_2(u_1))$ be the point of the curve $B(r)$ parameterized by u_1. It comes $M(u_1) = b(u_1)$. Define $\omega_1(u_1) = f_1(p(u_1), w_1(u_1))$ and $\omega_2(u_1) = f_2(p(u_1), w_2(u_1))$ and $\omega(u_1) = (\omega_1(u_1), \omega_2(u_1))$. Walras law for individual demands imply $w_1(u_1) = p(u_1) \cdot \omega_1(u_1)$ and $w_2 = p \cdot \omega_2(u_1)$. The pair $(p(u_1), \omega(u_1))$ is therefore an equilibrium and, actually, a no-trade equilibrium since $\omega_i(u_1) = f_i(p(u_1), p(u_1) \cdot \omega_i(u_1))$ for $i = 1, 2$. In addition, this equilibrium is regular since every no-trade equilibrium is regular by [1] or [5], Proposition 8.2.

The budget hyperplane $A(\omega(u_1))$ depends continuously on u_1. Therefore, the function $u_1 \to \delta(u_1) = \Delta(p(u_1), \omega(u_1))$ is also continuous.

The function $\delta(u_1)$ is different from 0 for all $u_1 \in [u_1(0), u_1(r)]$ since every no-trade equilibrium is regular. Therefore, it suffices to check the sign of this function

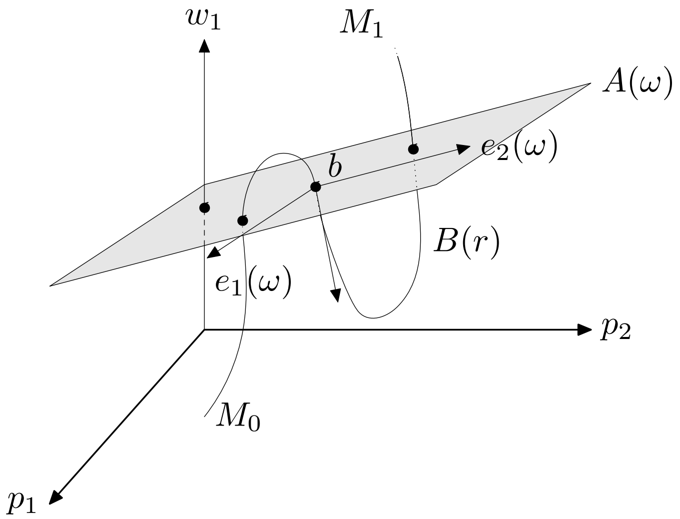

Figure 2: Orientation of $A(\omega)$ and $B(r)$ at b

for any particular value of u_1. A good candidate is $u_1 = u_1(0)$. Then, $\omega(u_1) = (0, r)$ and the vector $e_j(\omega(u_1))$ is equal to $(\pi_j, 0)$ for $1 \leq j \leq \ell - 1$. The point M_0 is in the horizontal hyperplane $A(0, r)$, a hyperplane with equation $w_1 = 0$ in the $(\bar{p}, w_1)$ coordinate system. Furthermore, the tangent vector $t(u_1)$ to the curve $B(r)$ at $u_1 = u_1(0)$ necessarily points upwards. This implies that the ℓ-th coordinate of the vector $t(u_1)$ is ≥ 0 for $u_1 = u_1(0)$ and cannot be equal to 0 because of the transversality property. This proves the strict inequality $\delta(u_1(0)) > 0$. $\square$

Figure 2 shows an example of a negative index number at the intersection point b of $A(\omega)$ and $B(r)$ for $\ell = 3$ goods.

Remark 3. Lemma 1 can be reformulated as saying that the index number of the regular equilibrium $(p, \omega) \in E$ is the same thing as the intersection number in the sense of [9], page 96 at the intersection $b = (p, p \cdot \omega_1, p \cdot \omega_2)$ (a point also denoted by $M(u_1)$ in the earlier sections) of the submanifolds $A(\omega)$ and $B(r)$.

Remark 4. If the endowment vector $\omega \in \Omega$ is regular (i.e., all equilibria $(p, \omega) \in E$ associated with ω are regular), there is only a finite number of equilibria [7]. It has been shown by Dierker that the sum of the indices over all these equilibria is an invariant equal to $+1$ [8]. This number is the same thing as the oriented intersection number of the submanifold $B(r)$ and $A(\omega)$ as defined in [9], page 107.

It follows from the value equal to $+1$ of the oriented intersection number that the number of equilibria of a regular economy is odd. With this number equal to $2n + 1$, $n + 1$ equilibria have an index equal to $+1$ and n have an index equal to -1.

4. Application to the transfer problem

The key issue is the relation between the transfer problem and the intersection number of $A(\omega)$ and $B(r)$ at their intersection point b corresponding to the regular equilibrium $(p, \omega) \in E$.

4.1. Giving away resources and budget hyperplanes

Giving away resources translates very nicely in terms of budget hyperplanes. By definition, the hyperplane $A(\omega')$ lies below the hyperplane $A(\omega)$ if the strict inequality $p \cdot \omega_1' < p \cdot \omega_1$ is satisfied for any $p \in S$. Note that the positions of $A(\omega)$ and $A(\omega')$ relative to each other are described only above the price set S. It then comes:

Lemma 2. *The hyperplane $A(\omega')$ lies below the hyperplane $A(\omega)$ if and only if $\omega_1' \lneqq \omega_1$.*

Proof. The condition is equivalent to $p \cdot (\omega_1' - \omega_1) < 0$ for any $p \in S$. This readily implies that all coordinates of $\omega_1' - \omega_1$ are ≤ 0 and one of them at least is strictly negative. □

4.2. The equilibrium selection map: geometric version

The concept of regular equilibrium is easily reformulated within the geometric setup. The equilibrium $(p, \omega) \in E$ is regular if the curve $B(r)$ and the hyperplane $A(\omega)$ intersects transversally at the point $b = (p, w_1, w_2)$ where $w_1 = p \cdot \omega_1$ and $w_2 = p \cdot \omega_2$. Let u_1 be consumer 1's utility level such that $M(u_1) = b$. The vector $t(u_1)$ is tangent to the curve $B(r)$ at the point b and, therefore, is not contained in the hyperplane $A(\omega)$ because of transversality.

Given some sufficiently small neighborhood of b, transversality at b implies that the intersection $B(r) \cap A(\omega')$ has a unique point b' in that neighborhood for ω' sufficiently close to ω. For ω' in that neighborhood, this construction defines a map $\omega' \to b' = b(\omega')$. In addition, it comes $b(\omega) = b$. The composition of that map with the projection $b' = (p', w_1', w_2') \to p' \in S$ is the (local) equilibrium price selection map of Section 2.5.

4.3. Index value of a regular equilibrium with a transfer problem

Theorem. *The regular equilibrium $(p, \omega) \in E$ features a transfer problem if and only if its index is equal to -1.*

Proof. Let $b(\omega) = (p, p \cdot \omega_1, p \cdot \omega_2) \in B(r) \cap B(\omega)$. Let $u_1 = u_1(f_1(p, p \cdot \omega_1))$. With the notation of earlier sections, it comes $b(\omega) = M(u_1)$. The arc $\widehat{M(u_1)\, M_1}$ (resp. $\widehat{M_0\, M(u_1)}$) out of the curve $B(r)$ consists of the points of $B(r)$ that are parameterized by utility levels $\geq u_1$ (resp. $\leq u_1$).

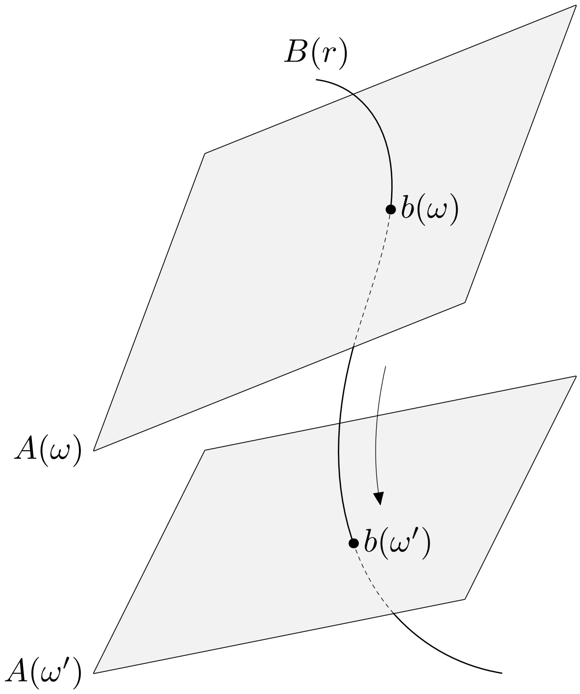

Figure 3: Orientation at the intersection and the transfer problem

If the intersection number of $B(r)$ and $A(\omega)$ at $b(\omega)$ is equal to -1, there is a neighborhood $V \subset H(r)$ of the point $b(\omega) = M(u_1)$ such that the points of the intersection $\widehat{M(u_1)\, M_1} \cap V$ are below the hyperplane $A(\omega)$. Similarly, the points of the intersection $\widehat{M_0\, M(u_1)} \cap V$ are above $A(\omega)$.

Let the open neighborhood U be the domain of the local equilibrium price selection map $s : U \to S$ of Section 2.5 defined by the regular equilibrium $(p, \omega) \in E$. For $\omega' \in U$ and $\omega' \lneqq \omega$, the hyperplane $A(\omega')$ is below $A(\omega)$. The point $b(\omega')$ is therefore below the hyperplane $A(\omega)$ and, therefore, belongs to the path $\widehat{M(u_1)\, M_1}$. Consumer 1's utility $u_1(f_1(s(\omega'), s(\omega') \cdot \omega'_1)$ is therefore strictly higher than $u_1 = u_1(f_1(s(\omega), s(\omega) \cdot \omega_1)$.

The same line of reasoning shows that if the intersection number of $B(r)$ and $A(\omega)$ at $b(\omega)$ is equal to $+1$, then the strict inequality

$$u_1(f_1(s(\omega'), s(\omega') \cdot \omega'_1) < u_1(f_1(s(\omega), s(\omega) \cdot \omega_1)$$

is satisfied for any $\omega' \in U$ with $\omega'_1 \lneqq \omega_1$. $\square$

The theorem proves the importance of the sets of regular equilibria with an index value equal to $+1$ and -1 respectively. Those sets partition the set of regular equilibria into pathconnected components. It is shown in [6] that the set of equilibria with an index value equal to $+1$ is pathconnected and contains the set of no-trade equilibria. This implies that the transfer problem can exist only for sufficiently large volumes of trade.

5. Concluding comments

By transferring resources from one country to another, the goal is generally to make the receiving country better off. It follows from the theorem of this paper that quirks in the market mechanism render that goal impossible to achieve at equilibria with an index number equal to -1. Samuelson's intuition was that the causes for this misbehavior of competitive markets were somehow related with those that create instability. This intuition is correct since an equilibrium with index number -1 is unstable. Having an index value equal to $+1$ does not imply stability, but at those equilibria, the behavior of competitive markets does not interfere with the goal of the donor to make the receiver better off. This makes the concept of index value equal to $+1$ a possible substitute to the concept of stability. This is obviously true for the case of two consumers. The general case of an arbitrary number of consumers justifies further research.

From the perspective of comparative statics, it is noteworthy that the set of regular equilibria with an index value equal to $+1$ is a pathconnected subset of the equilibrium manifold, a subset that also constrains the subset of no-trade equilibria.

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