On the stability of nonsunspot equilibria*

Yves Balasko†

July 12, 2009

Abstract

This paper analyzes the stability of the (Pareto efficient) nonsunspot equilibria as a function of the constraints faced by consumers in their ability to transfer wealth between states of nature. The nonsunspot equilibria are independent of the constraints since they can be identified to the equilibria of the associated certainty economy. It is shown that the equilibria that are stable for the certainty economy define nonsunspot equilibria that are stable in the following two polar cases: 1) All consumers are unconstrained; 2) All consumers are fully constrained. Furthermore, the stable certainty equilibria with small trade vectors define nonsunspot equilibria that are stable independently of the constraint levels. Instability can develop for intermediate constraint levels only at nonsunspot equilibria that feature sufficiently large trade vectors. A small change in the constraint levels may then trigger a jump from a Pareto efficient nonsunspot equilibrium to a Pareto inefficient sunspot equilibrium.

1. Introduction

The question of the existence of sunspot equilibria is well understood. When consumers can freely transfer wealth between states of nature, the only equilibria are the nonsunspot equilibria. The corresponding equilibrium allocations are Pareto efficient. To exist, sunspot equilibria require that some consumers are unable to transfer wealth across states of nature. The sunspot equilibria then coexist with the nonsunspot equilibria, the latter being independent of the constraints faced by the consumers. Not only are sunspot equilibrium allocations Pareto inefficient, their number increases from zero to a number that can be very large when all consumers are fully constrained: see [2, 5, 7].

The question is then whether a Pareto efficient nonsunspot equilibrium allocation can bifurcate into a Pareto inefficient sunspot equilibrium allocation as a consequence of a change in the constraints consumers are facing in the economy.

A remarkable property of nonsunspot equilibria is that consumers do not actually transfer wealth between states of nature. The insurance markets that would permit these wealth transfers are inactive. One might then expect that closing these markets would have no effect on the economy. I show that this intuition is misleading. There exist economies and constraint levels such that a change in the constraint levels may induce the economy to bifurcate from a Pareto efficient nonsunspot equilibrium allocation to a Pareto inefficient sunspot equilibrium allocation.

*The author is grateful to participants to a seminar at Cornell and to Karl Shell in particular for their valuable comments.

†Department of Economics and Related Studies. University of York, UK. Email: yb501@york.ac.uk

Bifurcations between different solutions of an equation (here, the equilibrium equation) occur whenever the equation depends on some variable parameters. The simplest form of transition happens when one solution is canceled out by another solution, which typically happens at critical values of the variable parameters. The process that selects one solution among the multiple solutions of the equation then has no other alternative than jumping from the solution that has just been canceled out to another solution. For example, this is what is observed in the general equilibrium model when the endowment vector crosses the set of singular economies: see [3], p. 23 and p. 111. This canceling out of equilibria cannot occur for the nonsunspot equilibria (of the sunspot model) when constraint levels are the only variable parameters because nonsunspot equilibria do not depend on these constraint levels. The sudden transition from a nonsunspot equilibrium to another equilibrium, possibly a sunspot equilibrium, requires a more subtle approach than just counting the number of equilibria of an economy.

The question of how the economy selects an equilibrium among several possible ones boils down to the specification of an appropriate price adjustment dynamics. In the case of the standard general equilibrium model with certainty, the most popular example of such dynamics is Walras tatonnement, its mathematical formulation being actually due to Samuelson [13]. But the original version of Walras tatonnement is plagued with several problems. One of them is the arbitrary choice of a numeraire, which creates an artificial asymmetry between the goods. Another one is the arbitrariness of the price adjustment speeds. The trouble is that the stability of an equilibrium often depends on the choice of the numeraire and adjustment speeds.

The symmetry issue becomes even more important here in the sunspot model where states of nature can be equiprobable, in which case they should be treated symmetrically. I therefore use in the current paper the reformulation of tatonnement considered for the general equilibrium model with certainty in [4]. Then, all goods are treated symmetrically and all adjustment speeds are determined endogenously. For details, see [4].

A first result of the current paper is the extension to stability of the property that certainty equilibria are immune to sunspots [2]. All stable certainty equilibria are shown to define stable nonsunspot equilibria in the fully unconstrained sunspot economy. All the other results deal with the relationships between stability, or lack of stability, of nonsunspot equilibria with the constraint levels of the sunspot economy. For example, the nonsunspot equilibria associated with stable certainty equilibria are also stable in the fully constrained sunspot economy just as in the fully unconstrained economy. But stability is not preserved for all equilibria and constraint levels. There exist stable certainty equilibria that define unstable nonsunspot equilibria for intermediate constraint levels in the sunspot economy. A property is identified in this paper under which a stable certainty equilibrium defines stable nonsunspot equilibria for all constraint levels. This property is satisfied in particular for certainty equilibria that carry small trade vectors. It is only the equilibria with large trade vectors that may become unstable for some intermediate values of the constraint levels.

The first part of this paper contains a short introduction to the theory of sunspot equilibria, which makes this paper as much self-contained as possible. On the mathematical side, only linear algebra and the definition of the stability of an equilibrium (or fixed point) of a dynamical system and its characterization through linearization are taken for granted. For references, see for example [10].

The paper is organized as follows. Section 2 describes the certainty economy associated with a sunspot economy. Section 3 deals with the sunspot economy proper and its relation with the certainty economy. Section 4 contains the main results of the paper. Section 5 is devoted to a few concluding comments. All mathematical proofs are gathered in an appendix.

2. The certainty economy

The study of sunspot and nonsunspot equilibria involves two closely related models: 1) A standard general equilibrium model with no uncertainty also known as the certainty economy; 2) The sunspot economy proper that is derived from the certainty economy by the introduction of uncertainty through a set of states of nature, additional assumptions being made regarding preferences, endowments and budget constraints.

2.1. The certainty economy

Goods and prices

There are ℓ goods and m consumers. All consumption sets are equal to the strictly positive orthant $X = \mathbb{R}^{\ell}_{++}$, i.e., every commodity is consumed. The consumption bundle of consumer i is represented by the vector $\bar{x}_i = (x_i^1, \ldots, x_i^{\ell}) \in X$. The price vector $\bar{p} = (p_1, \ldots, p_{\ell}) \in X$ has also all its components strictly positive and is *not normalized.*

Utility functions

For i varying from 1 to m, consumer i's preferences are represented by a utility function $u_i : X \to \mathbb{R}$ that satisfies the following properties that are standard in smooth equilibrium analysis: 1) Smoothness, i.e., differentiability up to any order; 2) All first order partial derivatives of u_i are strictly positive (monotonicity), a property which can be reformulated with the help of the gradient vector $Du_i(\bar{x}_i) \in X$ for $\bar{x}_i \in X$; 3) The Hessian matrix of the second order partial derivatives of u_i defines a negative definite quadratic form: $\bar{z}^T D^2 u_i(\bar{x}_i)\bar{z} \leq 0$ for the column matrix $\bar{z} \in \mathbb{R}^{\ell}$, the inequality being strict for $\bar{z} \neq 0$ (smooth concavity of the utility function); 4) Every indifference surface $\{\bar{y}_i \in X \mid u_i(\bar{y}_i) = u_i(\bar{x}_i)\}$ is closed in $\mathbb{R}^{\ell}$ for any $\bar{x}_i \in X$ (boundary condition that also implies that all goods are consumed).

Consumer's demand

Given the price vector $\bar{p} \in X$ and wealth $w_i > 0$, consumer i's demand maximizes the utility $u_i(\bar{x}_i)$ subject to the budget constraint $\bar{p} \cdot \bar{x}_i \leq w_i$.

Consumer i's demand is unique and is denoted by $\bar{x}_i = \bar{f}_i(\bar{p}, w_i)$. Consumer i's demand function $\bar{f}_i : X \times \mathbb{R}_{++} \to X$ is then homogeneous of degree zero, smooth, satisfies Walras law (i.e., the identity $\bar{p} \cdot \bar{f}_i(\bar{p}, w_i) = w_i$). We denote by $\bar{J}_i(\bar{p}, \bar{\omega}_i)$ the Jacobian matrix of the individual demand function $\bar{p} \to \bar{f}_i(\bar{p}, \bar{p} \cdot \bar{\omega}_i)$ at $\bar{p} \in X$. This matrix coincides with the Slutsky matrix for $\bar{\omega}_i = \bar{f}_i(\bar{p}, \bar{p} \cdot \bar{\omega}_i)$ and is then, symmetric, negative semidefinite, with only one eigenvalue equal to 0 with corresponding eigenvector the price vector $\bar{p} \in X$.

2.2. Initial endowments

In addition, consumer i, with $1 \leq i \leq m$, is endowed with the commodity bundle $\bar{\omega}_i \in X$ before the markets open.

Let $\bar{\boldsymbol{\omega}} = (\bar{\omega}_1, \ldots, \bar{\omega}_m) \in X^m$ denote the vector representing the endowments of the m consumers. This vector parameterizes the certainty economy, economy denoted by $\bar{\mathcal{E}}(\bar{\boldsymbol{\omega}})$ to illustrate its dependence on the endowment vector. The vector of total resources in the economy is denoted by $\bar{r} = \sum_i \bar{\omega}_i \in X$.

2.3. Equilibrium and aggregate excess demand functions

Aggregate excess demand

The aggregate excess demand in the certainty economy $\bar{\mathcal{E}}(\bar{\boldsymbol{\omega}})$ given the price vector $\bar{p} \in X$ is

$$\bar{z}(\bar{p}, \bar{\boldsymbol{\omega}}) = \left(\bar{f}_1(\bar{p}, \bar{p} \cdot \bar{\omega}_1) - \bar{\omega}_1\right) + \cdots + \left(\bar{f}_m(\bar{p}, \bar{p} \cdot \bar{\omega}_m) - \bar{\omega}_m\right).$$

The aggregate excess demand function of the certainty economy $\bar{\mathcal{E}}(\bar{\boldsymbol{\omega}})$ is the map $\bar{p} \to \bar{z}(\bar{p}, \bar{\boldsymbol{\omega}})$ from X into $\mathbb{R}^\ell$. This function is homogeneous of degree zero.

Equilibrium and equilibrium price vector

The price vector $\bar{p} \in X$ is an *equilibrium price vector* of the certainty economy $\bar{\mathcal{E}}(\bar{\boldsymbol{\omega}})$ if it is a zero of the aggregate demand function $\bar{z}(\bar{p}, \bar{\boldsymbol{\omega}})$.

The pair $(\bar{p}, \bar{\boldsymbol{\omega}})$ is an *equilibrium* if the price vector $\bar{p} \in X$ is an *equilibrium price vector* of the certainty economy $\bar{\mathcal{E}}(\bar{\boldsymbol{\omega}})$.

The *trade vector* associated with the equilibrium $(\bar{p}, \bar{\boldsymbol{\omega}})$ is the vector

$$\left(\bar{f}_1(\bar{p}, \bar{p} \cdot \bar{\omega}_1) - \bar{\omega}_1, \ldots, \bar{f}_m(\bar{p}, \bar{p} \cdot \bar{\omega}_m) - \bar{\omega}_m\right).$$

A *notrade equilibrium* is such $\bar{\omega}_i = \bar{f}_i(\bar{p}, \bar{p} \cdot \bar{\omega}_i)$ for $i = 1, 2, \ldots, m$.

Equilibrium allocation

The demand of consumer i for the price vector $\bar{p} \in X$ is equal to $\bar{x}_i = \bar{f}_i(\bar{p}, \bar{p} \cdot \bar{\omega}_i)$. When the price vector $\bar{p} \in X$ is an equilibrium price vector, the vector $\bar{\boldsymbol{x}} = (\bar{x}_1, \ldots, \bar{x}_m) \in X^m$ is feasible (i.e., $\sum_i \bar{x}_i = \sum_i \bar{\omega}_i = \bar{r}$) and $\bar{\boldsymbol{x}}$ is known as the *equilibrium allocation* of the certainty economy associated with the equilibrium price vector $\bar{p} \in X$.

Jacobian matrix of aggregate excess demand

Let $\bar{J}(\bar{p}, \bar{\boldsymbol{\omega}})$ denote the Jacobian matrix of the aggregate excess demand function for the certainty economy at the price vector $\bar{p} \in X$. It comes

$$\bar{J}(\bar{p}, \bar{\boldsymbol{\omega}}) = \sum_i \bar{J}_i(\bar{p}, \bar{\omega}_i). \tag{1}$$

At the notrade equilibrium $(\bar{p}, \bar{\boldsymbol{\omega}})$, the Jacobian matrix $\bar{J}(\bar{p}, \bar{\boldsymbol{\omega}})$ is symmetric, negative semidefinite, with only one eigenvalue equal to 0 and a corresponding eigenvector being the price vector $\bar{p} \in X$.

2.4. Price dynamics

Let the coordinatewise product $\square$ in $\mathbb{R}^\ell$ be defined by

$$x \square y = (x^1 y^1, \ldots, x^\ell y^\ell)$$

for $x = (x^1, \ldots, x^\ell)$ and $y = (y^1, \ldots, y^\ell)$ in $\mathbb{R}^\ell$.

The differential equation

The differential equation governing the evolution of the price vector $\bar{p}(t) \in X$ in the certainty economy $\bar{\mathcal{E}}(\bar{\boldsymbol{\omega}})$ is

$$\dot{\bar{p}}(t) \,\square\, \bar{r} = \bar{p}(t) \,\square\, \bar{z}\big(\bar{p}(t), \bar{\boldsymbol{\omega}}\big). \tag{2}$$

The total wealth $\bar{p}(t) \cdot \bar{r}$ does not depend on time t. The price vector $\bar{p}(t) \in X$ can then be normalized by setting this total wealth equal to one for example, i.e., $\bar{p} \cdot \bar{r} = 1$. Normalizing the price vector this way simplifies the definition and discussion of stability.

Stable (certainty) equilibria

By definition, the (normalized) equilibrium price vector $\bar{p}$ of the certainty economy $\bar{\mathcal{E}}(\bar{\boldsymbol{\omega}})$ is *stable* if it is a locally asymptotically stable fixed point of the dynamic system defined by differential equation (2) (on the set of normalized price vectors).

The following characterization of stability works for a category of equilibria that is sufficiently large for all practical purposes. By definition, the equilibrium $(\bar{p}, \bar{\boldsymbol{\omega}})$ is hyperbolic if the Jacobian matrix $\bar{J}(\bar{p}, \bar{\boldsymbol{\omega}})$ has only one eigenvalue equal to 0 and the real parts of the non-zero eigenvalues are different from 0. The set of hyperbolic equilibria is easily seen to be an open subset with full measure of the equilibrium manifold (i.e., the set of all equilibria $(\bar{p}, \bar{\boldsymbol{\omega}})$). Incidentally, the set of hyperbolic equilibria is contained in the set of regular equilibria. For details see [4].

Let Λ denote the $\ell \times \ell$ diagonal matrix with k-th coefficient p_k/r^k. The hyperbolic equilibrium $(\bar{p}, \bar{\boldsymbol{\omega}})$ is *stable* if and only if the matrix product $\bar{J}(\bar{p}, \bar{\boldsymbol{\omega}})\,\Lambda$ of the Jacobian matrix of aggregate excess demand at the equilibrium price vector $\bar{p} \in X$ and Λ has rank $\ell - 1$ (i.e., only one eigenvalue is equal to zero), the nonzero eigenvalues having strictly negative real parts. This characterization follows from the linearization of (2). For details, see [4].

2.5. Properties of the certainty economy

The following properties of the certainty economy $\bar{\mathcal{E}}(\bar{\boldsymbol{\omega}})$ are well-known: 1) Existence of an equilibrium price vector for any endowment vector $\bar{\boldsymbol{\omega}} \in X^m$; 2) Pareto efficiency of every equilibrium allocation; 3) The equilibrium price vector $\bar{p} \in X$ is stable if the Jacobian matrix of aggregate excess demand $\bar{J}(\bar{p}, \bar{\boldsymbol{\omega}})$ is such that $\bar{z}^T \bar{J}(\bar{p}, \bar{\boldsymbol{\omega}})\bar{z}$ is < 0 for the column matrix $\bar{z} \neq 0$ not collinear with the price vector $\bar{p} \in X$. (Matrix $\bar{J}(\bar{p}, \bar{\boldsymbol{\omega}})$ does not have to be symmetric; see [1].) This property of matrix $\bar{J}(\bar{p}, \bar{\boldsymbol{\omega}})$ is satisfied for small trade vectors

$$\big(\bar{f}_1(\bar{p}, \bar{p} \cdot \bar{\omega}_1) - \bar{\omega}_1, \dots, \bar{f}_m(\bar{p}, \bar{p} \cdot \bar{\omega}_m) - \bar{\omega}_m\big)$$

because the Jacobian matrix of aggregate excess demand $\bar{J}(\bar{p}, \bar{\boldsymbol{\omega}})$ is negative definite at every notrade equilibrium. For references, see [4] and [8].

3. The sunspot economy

Uncertainty is introduced by a finite number S of states of nature. State s has probability $\pi(s)$, with $1 \leq s \leq S$. The sunspot economy differs from a standard general equilibrium model with contingent goods ([8], Chapter 7) by several specific assumptions about preferences, endowments and consumers' budget constraints.

3.1. Preferences and endowments

The consumption space is the Cartesian product X^S. Let $\boldsymbol{x}_i = \big(x_i(1), \dots, x_i(S)\big) \in X^S$ and $\boldsymbol{p} = \big(p(1), \dots, p(S)\big) \in X^S$ denote a commodity bundle and a price vector respectively.

Utility functions

With the function $u_i : X \to \mathbb{R}$ defining the utility of consumer i in the certainty model, the utility of the commodity bundle $\boldsymbol{x}_i = (x_i(1), \dots, x_i(S)) \in X^S$ in the sunspot economy for the same consumer i is equal to the expected utility of $\boldsymbol{x}_i$, i.e,

$$v_i(\boldsymbol{x}_i) = \sum_{s=1}^{S} \pi(s) u_i(x_i(s)).$$

Endowments

In the sunspot economy, consumer i's endowment vector $\omega_i(s)$ does not depend on the states of nature:

$$\omega_i(1) = \cdots = \omega_i(S) = \bar{\omega}_i.$$

3.2. Constrained and unconstrained consumers

The sunspot economy can be viewed as made of a large number of replicas of the certainty economy where consumer i can be either constrained or unconstrained. When unconstrained, this consumer can freely transfer wealth between states of nature. When constrained, the same consumer faces a budget constraint for each state of nature.

The ratio of constrained consumers i to the total number of consumers i, constrained and unconstrained, is denoted by $\lambda_i \in [0, 1]$. The vector $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_m) \in [0, 1]^m$ represents the relative number of constrained consumers in the sunspot economy. The fully unconstrained case corresponds to $\boldsymbol{\lambda} = \boldsymbol{0} = (0, 0, \dots, 0)$ and the fully constrained case to $\boldsymbol{\lambda} = \boldsymbol{1} = (1, 1, \dots, 1)$.

Demand function of the unconstrained consumer

The *unconstrained consumer* i faces the unique budget constraint

$$\sum_{s=1}^{S} p(s) \cdot x_i(s) \leq \big(\sum_{s=1}^{S} p(s)\big) \cdot \bar{\omega}_i.$$

The demand function of an unconstrained consumer i is then a map $f_i : X^S \times \mathbb{R}_{++} \to X^S$ that is homogeneous of degree 0, smooth and satisfies Walras law.

Let $J_i(\boldsymbol{p}, \bar{\boldsymbol{\omega}}_i)$ denote the Jacobian matrix of demand function $\boldsymbol{p} \to f_i\big(\boldsymbol{p}, \boldsymbol{p} \cdot (\bar{\omega}_i, \dots, \bar{\omega}_i)\big)$. The Slutsky matrix of an unconstrained consumer is the Jacobian matrix $J_i(\boldsymbol{p}, \bar{\boldsymbol{\omega}}_i)$ for $\bar{\omega}_i = f_i\big(\boldsymbol{p}, \boldsymbol{p} \cdot (\bar{\omega}_i, \dots, \bar{\omega}_i)\big)$. The Slutsky matrix is symmetric semidefinite negative, of rank $S\ell - 1$, (i.e., one eigenvalue only is equal to 0), with the price vector $\boldsymbol{p} \in X^S$ being an eigenvector for that 0 eigenvalue.

Demand function of the constrained consumer

The *constrained consumer* i faces S budget constraints

$$p(s) \cdot x_i(s) \leq p(s) \cdot \bar{\omega}_i.$$

The demand function of a constrained consumer i is therefore the map $\left(\bar{f}_i\right)^S : (X \times \mathbb{R}_{++})^S \to X^S$ where $\bar{f}_i : X \times \mathbb{R}_{++} \to X$ is consumer i's demand function in the certainty economy. Note that the Slutsky matrix associated with the demand function of a constrained consumer i is symmetric semidefinite negative of rank $S(\ell - 1)$ (i.e., the eigenvalue 0 has an order of multiplicity equal to S).

The Jacobian matrix of the demand function $\boldsymbol{p} \to \left(\bar{f}_i(\bar{p}(s), \bar{p}(s) \cdot \bar{\omega}_i)\right)^S$ is then equal to the matrix

$$\begin{bmatrix} \bar{J}_i(\bar{p}(1), \bar{\omega}_i) & 0 & \dots & 0 \\ 0 & \bar{J}_i(\bar{p}(2), \bar{\omega}_i) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \bar{J}_i(\bar{p}(S), \bar{\omega}_i) \end{bmatrix}.$$

Remark 1. It follows from the monotonicity of the utility functions u_i that the budget constraints for both constrained and unconstrained consumers are binding. The inequality sign can therefore be replaced by the equality sign in the formulation of these constraints.

3.3. Individual and aggregate excess demands

Let $\mathcal{E}(\boldsymbol{\lambda}, \bar{\boldsymbol{\omega}})$ denote the sunspot economy associated with the constraint parameter $\boldsymbol{\lambda} \in [0, 1]^m$ and the endowment vector $\bar{\boldsymbol{\omega}} \in X^m$.

The *individual excess demand* of the constrained and unconstrained consumer i in the sunspot economy $\mathcal{E}(\boldsymbol{\lambda}, \bar{\boldsymbol{\omega}})$ is equal to

$$z_i(\boldsymbol{p}, \lambda_i, \bar{\omega}_i) = (1 - \lambda_i) f_i\big(\boldsymbol{p}, (\sum_{s=1}^{S} p(s)) \cdot \bar{\omega}_i\big) + \\ \lambda_i \big(\bar{f}_i(p(1), p(1) \cdot \bar{\omega}_i), \dots, \bar{f}_i(p(S), p(S) \cdot \bar{\omega}_i)\big) - (\bar{\omega}_i, \dots, \bar{\omega}_i).$$

The *aggregate excess demand* in the sunspot economy $\mathcal{E}(\boldsymbol{\lambda}, \bar{\boldsymbol{\omega}})$ for the price vector $\boldsymbol{p}$ is equal to

$$z(\boldsymbol{p}, \boldsymbol{\lambda}, \bar{\boldsymbol{\omega}}) = \sum_{i=1}^{m} z_i(\boldsymbol{p}, \lambda_i, \bar{\omega}_i).$$

3.4. Sunspot and nonsunspot equilibria

The price vector $\boldsymbol{p} = \big(p(1), \dots, p(S)\big) \in X^S$ is an equilibrium price vector of the sunspot economy $\mathcal{E}(\boldsymbol{\lambda}, \bar{\boldsymbol{\omega}})$ if the aggregate excess demand for that price vector is equal to zero:

$$z(\boldsymbol{p}, \boldsymbol{\lambda}, \bar{\boldsymbol{\omega}}) = 0.$$

Let $\boldsymbol{x} = \big(\boldsymbol{x}(1), \dots, \boldsymbol{x}(S)\big) \in (X^m)^S$ denote the (equilibrium) allocation associated with the equilibrium price vector $\boldsymbol{p} = \big(p(1), \dots, p(S)\big) \in X^S$. By definition, the price vector $\boldsymbol{p}$ is a *nonsunspot equilibrium* price vector if the corresponding equilibrium allocation is the same for all states of nature, i.e., $\boldsymbol{x}(1) = \cdots = \boldsymbol{x}(S)$. A *sunspot equilibrium allocation* is an equilibrium allocation that depends on the realization of the state of nature, i.e., there exists states of nature s and s', with $s \neq s'$, such that $\boldsymbol{x}(s) \neq \boldsymbol{x}(s')$. The corresponding equilibrium price vector is then a *sunspot equilibrium* price vector.

Relation between the nonsunspot equilibria of the sunspot economy and the equilibria of the certainty economy

Let now $\boldsymbol{p} = \big(p(1), \dots, p(S)\big) \in X^S$ be a nonsunspot equilibrium price vector. The (non normalized) price vectors $p(1), \dots, p(S)$ are then collinear. Any vector $\bar{p} \in X$ collinear with $p(1), \dots, p(S)$ is then an equilibrium price vector of the certainty economy $\bar{\mathcal{E}}(\bar{\boldsymbol{\omega}})$. Conversely, let $\bar{\boldsymbol{x}} \in X^m$ be the equilibrium allocation that is associated in the certainty economy $\bar{\mathcal{E}}(\bar{\boldsymbol{\omega}})$ with the price vector $\bar{p} \in X$. The allocation $\boldsymbol{x} = \big(\boldsymbol{x}(1), \dots, \boldsymbol{x}(S)\big) \in (X^m)^S$ where $\boldsymbol{x}(1) = \cdots = \boldsymbol{x}(S) = \bar{\boldsymbol{x}}$ is a nonsunspot equilibrium allocation. It is associated with the price vector $\boldsymbol{p} = \big(p(1), \dots, p(S)\big) \in X^S$ where $p(1) = \cdots = p(S) = \bar{p}$. For details, see [5].

3.5. Price dynamics in the sunspot model

The price dynamics for the sunspot economy $\mathcal{E}(\boldsymbol{\lambda}, \bar{\boldsymbol{\omega}})$ is similar to the one for the certainty economy $\bar{\mathcal{E}}(\bar{\boldsymbol{\omega}})$ except that now the number of goods is equal to $S\ell$ instead of ℓ. With the box notation now set in $(\mathbb{R}^\ell)^S$, the differential equation of the dynamical process in the sunspot economy takes the form

$$\dot{\boldsymbol{p}}(t) \,\square\, \boldsymbol{r} = \boldsymbol{p}(t) \,\square\, \boldsymbol{z}(\boldsymbol{p}(t), \boldsymbol{\lambda}, \bar{\boldsymbol{\omega}}) \tag{3}$$

where $\boldsymbol{r} = (\bar{r}, \dots, \bar{r})$ represents the total resources in the sunspot economy $\mathcal{E}(\boldsymbol{\lambda}, \bar{\boldsymbol{\omega}})$ with S states of nature.

The total wealth $\boldsymbol{p}(t) \cdot \boldsymbol{r}$ is constant and can be used to normalize the price vector $\boldsymbol{p}$. Here again, normalizing the price vector simplifies the definition and discussion of stability. To make the normalization consistent with the certainty economy, the price vector $\boldsymbol{p}$ is then normalized by the convention $\boldsymbol{p} \cdot \boldsymbol{r} = \big(p(1) + \cdots + p(S)\big) \cdot \bar{r} = S$, the number of states of nature.

3.6. Stability of equilibrium in the sunspot model

By definition, the normalized equilibrium price vector $\boldsymbol{p} = (p(1), \dots, p(S))$ of the sunspot economy $\mathcal{E}(\boldsymbol{\lambda}, \bar{\boldsymbol{\omega}})$ is *stable* if it is locally asymptotically stable for the dynamics defined by differential equation (3).

Let $J(\boldsymbol{p}, \boldsymbol{\lambda}, \bar{\boldsymbol{\omega}})$ denote the Jacobian matrix of aggregate excess demand at the equilibrium price vector $\boldsymbol{p} = (p(1), \dots, p(S)) \in X^S$. Let

$$L = \begin{bmatrix} \Lambda & 0 & \cdots & 0 \\ 0 & \Lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Lambda \end{bmatrix}.$$

The equilibrium $(\boldsymbol{p}, \bar{\boldsymbol{\omega}})$ is *hyperbolic* if only one eigenvalue of the matrix $J(\boldsymbol{p}, \boldsymbol{\lambda}, \bar{\boldsymbol{\omega}})\ L$ is equal to 0, the real parts of the other eigenvalues being different from 0. It follows from [4] that the set of hyperbolic equilibria is an open subset with full measure of the equilibrium manifold. This property enables us to limit the study of stability to hyperbolic equilibria for which we have the following characterization:

Proposition 1. *The (hyperbolic) equilibrium $(\boldsymbol{p}, \bar{\boldsymbol{\omega}})$ is stable if all nonzero eigenvalues of the product matrix $J(\boldsymbol{p}, \boldsymbol{\lambda}, \bar{\boldsymbol{\omega}})\ L$ have strictly negative real parts.*

3.7. The S-property or a stronger version of stability

The S-property for the nonsunspot equilibrium $(\boldsymbol{p}, \bar{\boldsymbol{\omega}})$ for the fully unconstrained sunspot economy $\mathcal{E}(\mathbf{0}, \bar{\boldsymbol{\omega}})$ is the property that the individual Jacobian matrix $J_i(\boldsymbol{p}, \mathbf{0}, \bar{\omega}_i)$ of every consumer i satisfies the inequality $z^T J_i(\boldsymbol{p}, \mathbf{0}, \bar{\omega}_i) z \leq 0$ for any column matrix $z \in \mathbb{R}^{\ell S}$, the inequality being strict for z not collinear with the price vector $\boldsymbol{p}$, only one eigenvalue being equal to 0. Note that the S-property does not imply that matrix $J_i(\boldsymbol{p}, \mathbf{0}, \bar{\omega}_i)$ is symmetric.

The set of nonsunspot equilibria satisfying the S-property contains the set of notrade equilibria because, at those equilibria, matrix $J_i(\boldsymbol{p}, \mathbf{0}, \bar{\omega}_i)$ is symmetric, negative semidefinite, only one eigenvalue being equal to 0 with associated eigenvector the price vector $\boldsymbol{p} \in X^S$.

The S-property is robust to perturbations. Therefore, the set of equilibria satisfying the S-property has a nonempty interior that contains the set of notrade equilibria. In other words, the S-property is satisfied by all nonsunspot equilibria with sufficiently small trade vectors.

The following Proposition shows us that the S-property implies stability:

Proposition 2. *The nonsunspot equilibrium $(\boldsymbol{p}, \bar{\boldsymbol{\omega}})$ that satisfies the S-property is stable in the fully unconstrained sunspot economy $\mathcal{E}(\mathbf{0}, \bar{\boldsymbol{\omega}})$ and the corresponding certainty equilibrium $(\bar{p}, \bar{\boldsymbol{\omega}})$ is also stable in the certainty economy $\bar{\mathcal{E}}(\bar{\boldsymbol{\omega}})$.*

These properties are to be extended in two different directions in the following section.

4. The results

Let us recall that the initial motivation for this paper is the immunity of certainty equilibria to sunspots [2]. That property states that the addition of the S states of nature to the certainty economy $\bar{\mathcal{E}}(\bar{\boldsymbol{\omega}})$ without the parallel addition of new constraints has no impact on the equilibrium allocations of the sunspot economy $\mathcal{E}(\mathbf{0}, \bar{\boldsymbol{\omega}})$. In other words, the fully unconstrained sunspot economy $\mathcal{E}(\mathbf{0}, \bar{\boldsymbol{\omega}})$ has no sunspot equilibria. A natural question in the direct line of this immunity property is therefore whether the introduction of sunspots has any impact on the adjustment price dynamics towards the nonsunspot equilibria. More specifically, do the equilibria that are stable for the certainty economy remain stable in the fully unconstrained sunspot economy? Even more relevant in view of potential applications is the same question for any constraint level $\boldsymbol{\lambda} \in [0, 1]^m$.

4.1. The fully unconstrained case

The following property is the direct extension to the adjustment price dynamics of the immunity of certainty equilibria to sunspots.

Theorem 1. *The (hyperbolic) nonsunspot equilibrium $(\boldsymbol{p}, \bar{\boldsymbol{\omega}})$ of the fully unconstrained sunspot economy $\mathcal{E}(\mathbf{0}, \bar{\boldsymbol{\omega}})$ is stable if and only if the equilibrium $(\bar{p}, \bar{\boldsymbol{\omega}})$ of the certainty economy $\bar{\mathcal{E}}(\bar{\boldsymbol{\omega}})$ is stable.*

Theorem 1 also extends part of Proposition 2 to stable equilibria.

4.2. The fully constrained case

The introduction of sunspots leads to a usually quite large number of sunspot equilibria in the fully constrained case. Extrapolating on this observation, one might expect that some stable certainty equilibria define nonsunspot equilibria that lose their stability property in the fully constrained case. The following property shows us that this is not the case.

Theorem 2. *The nonsunspot equilibrium $(\boldsymbol{p}, \bar{\boldsymbol{\omega}})$ is stable for the fully constrained sunspot economy $\mathcal{E}(\mathbf{1}, \bar{\boldsymbol{\omega}})$ if and only if the equilibrium $(\bar{p}, \bar{\boldsymbol{\omega}})$ of the certainty economy $\bar{\mathcal{E}}(\bar{\boldsymbol{\omega}})$ is stable.*

4.3. Intermediate cases: Sufficient condition for stability

A stable certainty equilibrium therefore defines a nonsunspot equilibrium that is stable for small and high constraint levels. If such equilibrium were stable for all constraint levels, a change in the constraints would not suffice in itself to trigger a transition from a Pareto efficient nonsunspot to a Pareto inefficient sunspot equilibrium. The following sufficient condition for stability to be independent of the constraint levels is therefore important.

Theorem 3. *The nonsunspot equilibrium $(\boldsymbol{p}, \bar{\boldsymbol{\omega}})$ satisfying the S-property is stable for the sunspot economy $\mathcal{E}(\boldsymbol{\lambda}, \bar{\boldsymbol{\omega}})$ for all constraint levels $\boldsymbol{\lambda} \in [0, 1]^m$.*

Theorem 3 extends Proposition 2 to arbitrary constraint levels.

Corollary 1. *The nonsunspot equilibrium $(\boldsymbol{p}, \bar{\boldsymbol{\omega}})$ of the sunspot economy $\mathcal{E}(\boldsymbol{\lambda}, \bar{\boldsymbol{\omega}})$ is stable for all constraint levels $\boldsymbol{\lambda} \in [0, 1]^m$ for sufficiently small trade vectors.*

This corollary gives us a sufficient condition for the stability of the nonsunspot equilibria for all constraint levels. Another implication of Corollary 1 is that the class of nonsunspot equilibria that are stable for all constraint levels has a non empty interior that contains the set of notrade equilibria.

4.4. Intermediate cases: Existence of unstable equilibria

At this point, one may even wonder whether a stable certainty equilibrium can turn into an unstable nonsunspot equilibrium for some intermediate constraint levels $\boldsymbol{\lambda} \in (0, 1)^m$. The answer is given by the following

Theorem 4. *There exist (hyperbolic) stable certainty equilibria $(\bar{p}, \bar{\boldsymbol{\omega}})$ such that the associated nonsunspot equilibrium $(\boldsymbol{p}, \bar{\boldsymbol{\omega}})$ of the sunspot economy $\mathcal{E}(\boldsymbol{\lambda}, \bar{\boldsymbol{\omega}})$ is unstable at some intermediate constraint levels (i.e, for some $\boldsymbol{\lambda} \in (0, 1)^m$).*

It follows from Corollary 1 that such certainty equilibrium $(\bar{p}, \bar{\boldsymbol{\omega}})$ must feature a large trade vector. The combination of Theorems 1, 4 and Corollary 2 implies that there are sunspot economies $\mathcal{E}(\boldsymbol{\lambda}, \bar{\boldsymbol{\omega}})$ with $\boldsymbol{\lambda} \in (0, 1)^m$ (i.e., for intermediate constraint levels) for which the simple fact for the constraint level $\boldsymbol{\lambda}$ of crossing some critical values, whether it is towards higher or lower constraint levels, may suffice to destabilize an otherwise stable nonsunspot equilibrium. It follows from Corollary 1 that the corresponding stable certainty equilibria must have large trade vectors. This is not in contradiction with the destabilizing impact of the intensity of trade as measured by the length of the trade vector. Indeed, there exist equilibria in the certainty economy with large trade vectors that are nevertheless stable. In other words, large trade vectors do not necessarily imply instability even if, loosely speaking, the probability of instability for an equilibrium that is chosen at random increases with the length of its trade vector.

5. Concluding comments

Starting from the fully unconstrained case, we have seen that increasing the restrictions that consumers are facing in their ability to transfer wealth between states of nature may generate enough instability in the sunspot economy to trigger a loss of Pareto efficiency. The theory therefore tells us that stability and efficiency losses can be observed with the implementation of restrictive monetary policies in an otherwise unconstrained economy, which fits rather well with the intuition.

But the theory also tells us that the story is far more complex. In particular, the property that the nonsunspot equilibria associated with stable certainty equilibria are also stable in the fully constrained case shows us that highly restrictive monetary policies that allow for little wealth transfers between states of nature have no or little destabilizing impact.

This highlights the theoretical importance of intermediate constraint levels. In addition to offering a much more relevant representation of real world economies, it follows from the results of this paper that any change in the values of the constraint levels, even a small one, may be sufficient to trigger a switch from stability to instability. How the stability of a Pareto efficient nonsunspot equilibrium fares under varying constraint levels cannot be predicted at the level of generality of this paper by just looking at the sense of variation of the constraints in the sunspot economy. For example, an even moderate expansionary monetary policy may very well end up destabilizing a Pareto efficient nonsunspot equilibrium.

To sum up, a Pareto efficient nonsunspot equilibrium that is stable in the certainty economy may be unstable for some intermediate values of the constraint levels of the sunspot economy. In such cases, the sunspot economy may endup selecting a non Pareto efficient sunspot equilibrium. This highlights the economic importance of the properties of sunspot equilibria even for setups where the only variable parameter in the economy is the constaint levels that economic agents are facing.

References

[1] K.J. Arrow and M. McManus. A note on dynamic stability. *Econometrica*, 26:448–454, 1958.

[2] Y. Balasko. Extrinsic uncertainty revisited. *Journal of Economic Theory*, 31:203–210, 1983.

[3] Y. Balasko. *Foundations of the Theory of General Equilibrium*. Academic Press, Boston, 1988.

[4] Y. Balasko. Out-of-equilibrium price dynamics. *Economic Theory*, 33:413–435, 2007.

[5] Y. Balasko, D. Cass, and K. Shell. Market participation and sunspot equilibria. *Review of Economic Studies*, 62:491–512, 1995.

[6] S. Bialas. A necessary and sufficient condition for the stability of convex combinations of stable polynomials or matrices. *Bulletin of the Polish Academy of Sciences*, 33:473–480, 1985.

[7] D. Cass and K. Shell. Do sunspots matter? *Journal of Political Economy*, 91:193–227, 1983.

[8] G. Debreu. *Theory of Value*. Wiley, New York, 1959.

[9] D. Hershkowitz. On cones and stability. *Linear Algebra and Its Applications*, 275/276:249–259, 1998.

[10] M.W. Hirsch and S. Smale. *Differential Equations, Dynamical Systems, and Linear Algebra*. Academic Press, New York, 1974.

[11] L. Hogben, R.A. Brualdi, A. Greenbaum, and R. Mathias. *Handbook of Linear Algebra*. CRC Press, 2007.

[12] C. Prechac. On regular symmetric economies. *Journal of Economic Theory*, 69:198–204, 1996.

[13] P.A. Samuelson. The stability of equilibrium. *Econometrica*, 9:97–120, 1941.

A. Proofs

This appendix is devoted to proofs. Following [2], it suffices to consider equiprobable states of nature, a continuity argument doing the rest. To simplify the notation, the number of states of nature S is assumed to be equal to two, the extension to an arbitrary number of states being straightforward. Let α and β denote the two states of nature, with probability $\pi(\alpha) = \pi(\beta) = .5$.

As seen in Section 3.6, the stability properties of the nonsunspot equilibrium $\big((\bar{p}, \bar{p}), \bar{\boldsymbol{\omega}}\big)$ of the economy $\mathcal{E}(\boldsymbol{\lambda}, \bar{\boldsymbol{\omega}})$ depend on the eigenvalues of the product of the Jacobian matrix of aggregate excess demand $J\big((\bar{p}, \bar{p}), \bar{\boldsymbol{\omega}}\big)$ with the diagonal matrix L. The results of this paper hinge on the relations between the two matrices $\bar{J}(\bar{p}, \bar{\boldsymbol{\omega}})\,\Lambda$ and $J\big((\bar{p}, \bar{p}), \bar{\boldsymbol{\omega}}\big)\,L$ and, more specifically, their eigenvalues.

A.1. Jacobian matrix of the fully unconstrained consumer i's demand at arbitrary $\bar{\omega}_i$

Lemma 1. *The Jacobian matrix of consumer i's demand $f_i\big((p(\alpha), p(\beta)), (p(\alpha) + p(\beta)) \cdot \bar{\omega}_i\big)$ at the price vector $\boldsymbol{p} = (\bar{p}, \bar{p})$ for the arbitrary endowment vector $\bar{\omega}_i$ takes the form*

$$\frac{Df_i}{D(p(\alpha), p(\beta))}\big((p(\alpha), p(\beta)), (p(\alpha) + p(\beta)) \cdot \bar{\omega}_i\big) = \begin{bmatrix} A_i & B_i \\ B_i & A_i \end{bmatrix} \tag{4}$$

Proof. The derivative of demand $f_i\big((p(\alpha), p(\beta)), (p(\alpha) + p(\beta)) \cdot \bar{\omega}_i\big)$ of consumer i as a function of the price vector $\boldsymbol{p} = \big(p(\alpha), p(\beta)\big)$ is equal to

$$\begin{bmatrix} \dfrac{Df_i^\alpha}{Dp(\alpha)} & \dfrac{Df_i^\alpha}{Dp(\beta)} \\[2ex] \dfrac{Df_i^\beta}{Dp(\alpha)} & \dfrac{Df_i^\beta}{Dp(\beta)} \end{bmatrix}$$

where $f_i = (f_i^\alpha, f_i^\beta) \in X^2$ denotes consumer i's demand for states of nature α and β.

The (h, k) term of matrix $\dfrac{Df_i^\alpha}{Dp(\alpha)}$ is equal to

$$\frac{df_i^{\alpha h}}{dp_k(\alpha)} = \frac{\partial f_i^{\alpha h}}{\partial p_k(\alpha)} + \frac{\partial f_i^{\alpha h}}{\partial w_i}\omega_i^k \tag{5}$$

with a similar formula for the (h, k) term of matrix $\dfrac{Df_i^\beta}{Dp(\beta)}$.

For $p(\alpha) = p(\beta) = \bar{p}$, an obvious symmetry argument implies the equalities

$$\frac{\partial f_i^{\alpha h}}{\partial p_k(\alpha)} = \frac{\partial f_i^{\beta h}}{\partial p_k(\beta)} \quad \text{and} \quad \frac{\partial f_i^{\alpha h}}{\partial w_i} = \frac{\partial f_i^{\beta h}}{\partial w_i}, \tag{6}$$

from which follows the equality

$$\frac{Df_i^\alpha}{Dp(\alpha)} = \frac{Df_i^\beta}{Dp(\beta)} = A_i.$$

A similar argument proves the equality

$$\frac{Df_i^\alpha}{Dp(\beta)} = \frac{Df_i^\beta}{Dp(\alpha)} = B_i,$$

which leads us to expression (4). □

Lemma 2. *The matrix $A_i + B_i$ is the Jacobian matrix of consumer i's demand $\bar{f}_i(\bar{p}, \bar{p}\cdot\bar{\omega}_i)$ in the certainty economy:*

$$\frac{D\bar{f}_i}{D\bar{p}}(\bar{p}, \bar{p}\cdot\bar{\omega}_i) = A_i + B_i \tag{7}$$

Proof. In order to prove expression (7), let us take the derivative of the first of these two equalities

$$\bar{f}_i(\bar{p}, \bar{p}\cdot\bar{\omega}_i) = f_i^\alpha\big((\bar{p},\bar{p}), (\bar{p}+\bar{p})\cdot\bar{\omega}_i\big) = f_i^\beta\big((\bar{p},\bar{p}), (\bar{p}+\bar{p})\cdot\bar{\omega}_i\big),$$

and apply the chain rule, which yields

$$\frac{D\bar{f}_i}{D\bar{p}} = \frac{Df_i^\alpha}{Dp(\alpha)} + \frac{Df_i^\alpha}{Dp(\beta)} = A_i + B_i.$$

□

Lemma 3. *The matrix $A_i - B_i$ is symmetric negative definite. In addition, this matrix depends only on $\bar{p} \in X$ and on $\bar{p}\cdot\bar{\omega}_i = w_i$.*

Proof. The (h, k) term of $A_i - B_i$ is equal to

$$\left(\frac{\partial f_i^{\alpha h}}{\partial p_k(\alpha)} + \frac{\partial f_i^{\alpha h}}{\partial w_i}\omega_i^k\right) - \left(\frac{\partial f_i^{\alpha h}}{\partial p_k(\beta)} + \frac{\partial f_i^{\alpha h}}{\partial w_i}\omega_i^k\right) = \frac{\partial f_i^{\alpha h}}{\partial p_k(\alpha)} - \frac{\partial f_i^{\alpha h}}{\partial p_k(\beta)}$$

which proves that $A_i - B_i$ depends only on $\bar{p} \in X$ and on $\bar{p}\cdot\bar{\omega}_i = w_i$.

This implies that $A_i - B_i$ can be computed for any $\bar{\omega}_i$ satisfying $\bar{p}\cdot\bar{\omega}_i = w_i$, which we do in the following Section. Then, symmetry and negative definiteness will follow from Lemma 6.

□

A.2. Jacobian matrix of the fully unconstrained consumer i's demand in the special case $\bar{\omega}_i = f_i(\bar{p}, \bar{p}\cdot\bar{\omega}_i)$

Lemma 4. *The Jacobian matrix of consumer i's demand $f_i\big((p(\alpha), p(\beta)), (p(\alpha)+p(\beta))\cdot\bar{\omega}_i\big)$ at the price vector $\boldsymbol{p} = (\bar{p}, \bar{p})$ for $\bar{\omega}_i = f_i(\bar{p}, \bar{p}\cdot\bar{\omega}_i)$ takes the form*

$$\frac{Df_i}{D(p(\alpha), p(\beta))}\big((p(\alpha), p(\beta)), (p(\alpha)+p(\beta))\cdot\bar{\omega}_i\big) = \begin{bmatrix} A_i^* & B_i^* \\ B_i^* & A_i^* \end{bmatrix} \tag{8}$$

that is symmetric, with one eigenvalue equal to zero, an eigenvector being the price vector $\boldsymbol{p} = (\bar{p}, \bar{p})$, all other eigenvalues being strictly negative.

Proof. The first part is just Lemma 1 applied to the special case $\bar{\omega}_i = f_i(\bar{p}, \bar{p}\cdot\bar{\omega}_i)$. The second part of the lemma follows from the observation that the Jacobian matrix is by definition a Slutsky matrix, and that the properties are those of the Slutsky matrix when the utility function is smoothly strictly concave.

□

Lemma 5. *The Jacobian matrix of consumer i's demand $\bar{f}_i(\bar{p}, \bar{p}\cdot\bar{\omega}_i)$ in the certainty economy for $\bar{\omega}_i = f_i(\bar{p}, \bar{p}\cdot\bar{\omega}_i)$ takes the form*

$$\frac{D\bar{f}_i}{D\bar{p}}(\bar{p}, \bar{p}\cdot\bar{\omega}_i) = A_i^* + B_i^*, \tag{9}$$

is symmetric, with one eigenvalue equal to zero, a corresponding eigenvector being the price vector $\bar{p}$, all other eigenvalues being strictly negative.

Proof. Again, it suffices to remark that these are just the properties of Slutsky matrices. □

Lemma 6. *The difference $A_i^* - B_i^*$ is symmetric negative definite.*

Proof. It follows from Lemma 4 and 5 that A_i^* and $A_i^* + B_i^*$ are symmetric, hence B_i^* symmetric. The difference $A_i^* - B_i^*$ is therefore symmetric.

Let I denote the $\ell \times \ell$ identity matrix. We have

$$\begin{bmatrix} I & I \\ I & -I \end{bmatrix} \begin{bmatrix} A_i^* & B_i^* \\ B_i^* & A_i^* \end{bmatrix} \begin{bmatrix} I & I \\ I & -I \end{bmatrix} = 2 \begin{bmatrix} A_i^* + B_i^* & 0 \\ 0 & A_i^* - B_i^* \end{bmatrix}. \tag{10}$$

This implies that the eigenvalues of matrices

$$\begin{bmatrix} A_i^* & B_i^* \\ B_i^* & A_i^* \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} A_i^* + B_i^* & 0 \\ 0 & A_i^* - B_i^* \end{bmatrix}$$

are the same. One and only one of these eigenvalues is equal to 0. Since 0 is an eigenvalue of $A_i^* + B_i^*$, this implies that all the eigenvalues of $A_i^* - B_i^*$ are strictly negative. □

A.3. Jacobian matrix of consumer i's excess demand in the sunspot economy $\mathcal{E}(\boldsymbol{\lambda}, \bar{\boldsymbol{\omega}})$ for arbitrary constraint level $\boldsymbol{\lambda} \in [0, 1]^m$

Lemma 7. *With A_i and B_i defined in Lemma 1 for $\bar{\omega}_i$ arbitrary, the Jacobian matrix of consumer i's demand facing the constraint level $\lambda_i \in [0, 1]$ at $(p(\alpha), p(\beta)) = (\bar{p}, \bar{p})$ takes the form*

$$\begin{bmatrix} A_i + \lambda_i B_i & (1 - \lambda_i) B_i \\ (1 - \lambda_i) B_i & A_i + \lambda_i B_i \end{bmatrix}$$

Proof. This Jacobian matrix is equal to

$$(1 - \lambda_i) \frac{Df_i}{D(p(\alpha), p(\beta))} \big((p(\alpha), p(\beta)), (p(\alpha) + p(\beta)) \cdot \bar{\omega}_i\big) + \\ \lambda_i \begin{bmatrix} \dfrac{D\bar{f}_i}{D\bar{p}}(p(\alpha), p(\alpha) \cdot \bar{\omega}_i) & 0 \\ 0 & \dfrac{D\bar{f}_i}{D\bar{p}}(p(\beta), p(\beta) \cdot \bar{\omega}_i) \end{bmatrix}, \tag{11}$$

expression which, by Lemma (1), is equal to

$$(1 - \lambda_i) \begin{bmatrix} A_i & B_i \\ B_i & A_i \end{bmatrix} + \lambda_i \begin{bmatrix} A_i + B_i & 0 \\ 0 & A_i + B_i \end{bmatrix}, \tag{12}$$

hence to

$$\begin{bmatrix} A_i & B_i \\ B_i & A_i \end{bmatrix} + \lambda_i \begin{bmatrix} B_i & -B_i \\ -B_i & B_i \end{bmatrix} = \begin{bmatrix} A_i + \lambda_i B_i & (1 - \lambda_i) B_i \\ (1 - \lambda_i) B_i & A_i + \lambda_i B_i \end{bmatrix}.$$

□

A.4. Jacobian matrix of aggregate excess demand $J\big((\bar{p}, \bar{p}), \boldsymbol{\lambda}, \bar{\boldsymbol{\omega}}, \big)$ in the sunspot economy $\mathcal{E}(\boldsymbol{\lambda}, \bar{\boldsymbol{\omega}})$

Lemma 8. *The Jacobian matrix $J\big((\bar{p}, \bar{p}), \boldsymbol{\lambda}, \bar{\boldsymbol{\omega}}\big)$ is made of four blocks*

$$J\big((\bar{p}, \bar{p}), \boldsymbol{\lambda}, \bar{\boldsymbol{\omega}}\big) = \begin{bmatrix} A(\boldsymbol{\lambda}) & B(\boldsymbol{\lambda}) \\ B(\boldsymbol{\lambda}) & A(\boldsymbol{\lambda}) \end{bmatrix}.$$

where

$$A(\boldsymbol{\lambda}) = \sum_i A_i + \sum_i \lambda_i B_i \quad , \quad B(\boldsymbol{\lambda}) = \sum_i (1 - \lambda_i) B_i.$$

Proof. The Jacobian matrix $J\big((\bar{p}, \bar{p}), \boldsymbol{\lambda}, \bar{\boldsymbol{\omega}}\big)$ is the sum of the Jacobian matrices of individual demand:

$$J\big((\bar{p}, \bar{p}), \bar{\boldsymbol{\omega}}, \boldsymbol{\lambda}\big) = \sum_i \begin{bmatrix} A_i + \lambda_i B_i & (1 - \lambda_i) B_i \\ (1 - \lambda_i) B_i & A_i + \lambda_i B_i \end{bmatrix},$$

matrices that all have the block structure of the Lemma. Therefore their sum also has the same block structure. Straightforward computations then yield the formulas defining $A(\boldsymbol{\lambda})$ and $B(\boldsymbol{\lambda})$.

□

Remark 2. An alternative proof of Lemma 8 for the special case of fully unconstrained sunspot economies, (i.e., $\boldsymbol{\lambda} = \mathbf{0}$) can be found in [12], Proposition 3.3.

Lemma 9.

$$A(\boldsymbol{\lambda}) + B(\boldsymbol{\lambda}) = \bar{J}(\bar{p}, \bar{\boldsymbol{\omega}}),$$
$$A(\boldsymbol{\lambda}) - B(\boldsymbol{\lambda}) = A(\mathbf{0}) - B(\mathbf{0}) + 2\sum_i \lambda_i B_i,$$

where $A(\mathbf{0}) - B(\mathbf{0})$ is symmetric negative definite.

Proof. It follows from the expressions of $A(\boldsymbol{\lambda})$ and $B(\boldsymbol{\lambda})$ that we have

$$A(\boldsymbol{\lambda}) + B(\boldsymbol{\lambda}) = \sum_i (A_i + B_i) = \bar{J}(\bar{p}, \bar{\boldsymbol{\omega}}).$$

Similarly, we have

$$A(\boldsymbol{\lambda}) - B(\boldsymbol{\lambda}) = \sum_i (A_i - B_i) + 2\sum_i \lambda_i B_i.$$

To conclude, it suffices to observe that $\sum_i (A_i - B_i)$ is equal to $A(\mathbf{0}) - B(\mathbf{0})$ and that each term $A_i - B_i$ is a symmetric negative definite matrix.

□

Lemma 10. *The Jacobian matrix of aggregate excess demand $J\big((\bar{p}, \bar{p}), \boldsymbol{\lambda}, \bar{\boldsymbol{\omega}}\big)$ is equivalent to the matrix*

$$\begin{bmatrix} \bar{J}(\bar{p}, \bar{\boldsymbol{\omega}}) & 0 \\ 0 & A(\mathbf{0}) - B(\mathbf{0}) + 2\sum_i \lambda_i B_i \end{bmatrix}$$

Proof. Straightforward computations show us

$$\begin{bmatrix} I & I \\ I & -I \end{bmatrix} \begin{bmatrix} A(\boldsymbol{\lambda}) & B(\boldsymbol{\lambda}) \\ B(\boldsymbol{\lambda}) & A(\boldsymbol{\lambda}) \end{bmatrix} \begin{bmatrix} I & I \\ I & -I \end{bmatrix} = 2 \begin{bmatrix} A(\boldsymbol{\lambda}) + B(\boldsymbol{\lambda}) & 0 \\ 0 & A(\boldsymbol{\lambda}) - B(\boldsymbol{\lambda}) \end{bmatrix}.$$

It then suffices to substitute to $A(\boldsymbol{\lambda})$ and $B(\boldsymbol{\lambda})$ their expressions in Lemma 9.

□

Lemma 11. *The eigenvalues of $J\big((\bar{p}, \bar{p}), \boldsymbol{\lambda}, \bar{\boldsymbol{\omega}}\big)$ consist of the eigenvalues of $\bar{J}(\bar{p}, \bar{\boldsymbol{\omega}})$ and $A(\mathbf{0}) - B(\mathbf{0}) + 2\sum_i \lambda_i B_i$*

Proof. Direct consequence of Lemma 10.

□

A.5. Matrices $J(\boldsymbol{p}, \boldsymbol{\lambda}, \bar{\boldsymbol{\omega}})\, L$ and $\bar{J}(\bar{p}, \bar{\boldsymbol{\omega}})\, \Lambda$

Lemma 12. *Matrix $J(\boldsymbol{p}, \boldsymbol{\lambda}, \bar{\boldsymbol{\omega}})\, L$ is made of the four blocks*

$$J\big((\bar{p}, \bar{p}), \boldsymbol{\lambda}, \bar{\boldsymbol{\omega}}\big)\, L = \begin{bmatrix} A(\boldsymbol{\lambda})\, \Lambda & B(\boldsymbol{\lambda})\, \Lambda \\ B(\boldsymbol{\lambda})\, \Lambda & A(\boldsymbol{\lambda})\, \Lambda \end{bmatrix}.$$

Proof. Obvious. □

Lemma 13. *The eigenvalues of matrix $J\big((\bar{p}, \bar{p}), \boldsymbol{\lambda}, \bar{\boldsymbol{\omega}}\big)\, L$ consist of the eigenvalues of $\bar{J}(\bar{p}, \bar{\boldsymbol{\omega}})\, \Lambda$ and $\big(A(\mathbf{0}) - B(\mathbf{0}) + 2\sum_i \lambda_i B_i\big)\, \Lambda$.*

Proof. It follows from Lemma 12 that the matrix product $J\big((\bar{p}, \bar{p}), \boldsymbol{\lambda}, \bar{\boldsymbol{\omega}}\big)\, L$ has the same block structure as the matrices in Lemma 10. It then suffices to reproduce the line of reasoning of Lemma 11.

□

A.6. Proofs of the propositions in the main text

Proof of Proposition 1. Follows readily from the definition of stability applied to the linearized version of the adjustment dynamics. See also [4] ☐

Proof of Proposition 2. We first prove that the nonsunspot equilibrium $\big((\bar{p}, \bar{p}), \bar{\boldsymbol{\omega}}\big)$ is stable. By the S-property the matrix

$$J_i\big((\bar{p}, \bar{p}), \mathbf{0}, \bar{\boldsymbol{\omega}}\big) = \begin{bmatrix} A_i & B_i \\ B_i & A_i \end{bmatrix}$$

is, for every consumer i, negative semidefinite, the eigenvalue 0 having multiplicity equal to 1, with associated eigenvector $(\bar{p}, \bar{p})$.

The sum $J\big((\bar{p}, \bar{p}), \mathbf{0}, \bar{\boldsymbol{\omega}}\big)$ therefore satisfies the same properties. Restricting the analysis to the hyperplane perpendicular to the price vector $(\bar{p}, \bar{p})$, we can apply readily to matrix $J\big((\bar{p}, \bar{p}), \mathbf{0}, \bar{\boldsymbol{\omega}}\big)$ the property that a negative definite matrix (though not necessarily symmetric) is diagonally stable, i.e., stable after multiplication by an arbitrary strictly positive diagonal matrix, here the matrix L. (See for example [1].)

We now prove that the certainty equilibrium $(\bar{p}, \bar{\boldsymbol{\omega}})$ is stable. The quadratic expression

$$[\bar{z}^T, \bar{z}^T)] \begin{bmatrix} A_i & B_i \\ B_i & A_i \end{bmatrix} \begin{bmatrix} \bar{z} \\ \bar{z} \end{bmatrix} = 2\,\bar{z}^T (A_i + B_i)\,\bar{z}$$

is ≤ 0 for all $\bar{z} \in \mathbb{R}^\ell$ not collinear with the price vector $\bar{p}$, which proves that $A_i + B_i$ is negative semidefinite with the price vector $\bar{p}$, an eigenvector associated with the eigenvalue 0 with multiplicity equal to 1. The matrix $A + B = \sum_i (A_i + B_i)$ then satisfies the same properties. It then suffices to reproduce the line of reasoning of the first part of this proof for matrix $A + B$ to get that the non zero eigenvalues of matrix $\bar{J}(\bar{p}, \bar{\boldsymbol{\omega}})\,\Lambda$ have strictly negative real parts.

☐

A.7. Proofs of the theorems in the main text

These proofs now require only a few lines at most.

Proof of Theorem 1. Matrix $A(\mathbf{0}) - B(\mathbf{0})$ is negative definite by Lemma 10. This matrix is therefore diagonally stable, i.e., all eigenvalues of matrix $A(\mathbf{0}) - B(\mathbf{0})\,\Lambda$ have strictly negative real parts. It then suffices to apply Lemma 13. ☐

Proof of Theorem 2. For $\boldsymbol{\lambda} = \mathbf{1}$, we have

$$A(\mathbf{1}) - B(\mathbf{1}) = A(\mathbf{0}) - B(\mathbf{0}) + 2\sum_i B_i = A(\mathbf{0}) + B(\mathbf{0} = \bar{J}(\bar{p}, \bar{\boldsymbol{\omega}}).$$

We then conclude by Lemma 13 again. ☐

Proof of Theorem 3. It follows from Theorem 2 that we can assume $\boldsymbol{\lambda} \neq \mathbf{1} = (1, 1, \ldots, 1)$.We rewrite $A(\boldsymbol{\lambda}) - B(\boldsymbol{\lambda})$ as

$$A(\boldsymbol{\lambda}) - B(\boldsymbol{\lambda}) = \sum_i \big((1 - \lambda_i)(A_i - B_i) + \lambda_i(A_i + B_i)\big). \tag{13}$$

By the S-property, matrix $A_i + B_i$ is negative semidefinite (not necessarily symmetric). Matrix $A_i - B_i = A_i^* - B_i^*$ is negative definite by Lemma 6. From $\boldsymbol{\lambda} \neq \mathbf{1} = (1, 1, \ldots, 1)$, matrix $A(\boldsymbol{\lambda}) - B(\boldsymbol{\lambda})$ is the sum of negative semidefinite matrices, one of them at least being negative definite. Their sum is therefore negative definite. Matrix $A(\boldsymbol{\lambda}) - B(\boldsymbol{\lambda})$ is therefore diagonally stable. The eigenvalues of $(A(\boldsymbol{\lambda}) - B(\boldsymbol{\lambda}))\,\Lambda$ all have strictly negative real parts. We then conclude by Lemma 13 again.

☐

Proof of Theorem 4. In formula (13), matrix $(A(\boldsymbol{\lambda}) - B(\boldsymbol{\lambda}))\,\Lambda$ is a linear combination of matrices $(A_i - B_i))\,\Lambda$ that are stable and of matrices $(A_i + B_i)\,\Lambda$ that are not necessarily stable even if their sum over all consumers $\big(\sum_i (A_i + B_i)\big)\,\Lambda$ is stable.

The set of stable matrices is not convex. (For references, see [6, 9] and [11].) It is easy to find examples of a convex combination $(1 - \lambda)H + \lambda K$ where matrix H is symmetric negative definite and matrix K stable such that, for some intermediate value $\lambda \in (0, 1)$, the convex combination fails to be stable. This proves the existence of unstable matrices already in the case of one consumer (i.e., $m = 1$) with speeds of adjustment defined by a matrix $\Lambda = I$, the identity matrix.

☐

Remark 3. It follows from these proofs that the precise value of the diagonal matrix Λ is inessential. In other words, our results would also hold for arbitrary speeds of adjustment and, therefore, also for the standard version of tatonnement.